

# Decay Estimates for Isentropic Compressible Navier-Stokes Equations in Bounded Domain

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## Abstract

In this paper, under the hypothesis that  $\rho$  is upper bounded, we construct a Lyapunov functional for the multidimensional isentropic compressible Navier-Stokes equations and show that the weak solutions decay exponentially to the equilibrium state in  $L^2$  norm. This can be regarded as a generalization of Matsumura and Nishida's results in [23], since our analysis is done in the framework of Lions [20] and Feireisl et al. [9], the higher regularity of  $(\rho, u)$  and the uniformly positive lower bound of  $\rho$  are not necessary in our analysis and vacuum may be admitted. Indeed, the upper bound of the density  $\rho$  plays the essential role in our proof.

**Keywords:** Compressible Navier-Stokes equations; decay estimates

## 1 Introduction

This paper is devoted to the asymptotic behavior of the solutions to the Navier-Stokes equations of an isentropic compressible fluid:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u, \end{cases} \quad (1.1)$$

where the density  $\rho = \rho(t, x)$  and the velocity  $u = (u^1(t, x), u^2(t, x), \dots, u^N(t, x))$  are functions of the time  $t \in (0, \infty)$  and the spatial coordinate  $x \in \Omega$  where  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 2$ , is a bounded regular domain.  $P(\rho) = a\rho^\gamma$  is the pressure, with  $a > 0$  and  $\gamma > 1$  being two positive constants. Since the constant  $a$  does not play any role in the analysis, we assume henceforth that  $a = 1$ . The constants  $\mu$  and  $\lambda$  are viscosity coefficients satisfying

$$\mu > 0, \quad \lambda + \frac{2}{N}\mu \geq 0.$$

We prescribe the initial conditions for the density and momenta:

$$\rho(0) = \rho_0, \quad (\rho u)(0) = m_0, \quad (1.2)$$

together with the no-slip boundary conditions for the velocity:

$$u|_{\partial\Omega} = 0. \quad (1.3)$$

Some of the previous works in this direction can be summarized as follows. The first general result on weak solutions to the multidimensional isentropic compressible Navier-Stokes equations with large initial data was obtained by Lions in [20], in which he used the renormalization skills introduced by DiPerna and Lions in [5] to obtain global weak solutions provided that the specific heat ratio  $\gamma$  is appropriately large, for example,  $\gamma \geq 3N/(N+2)$ ,  $N = 2, 3$ . Later, Feireisl, Novotný and Petzeltový [9] improved Lions's result to the case  $\gamma > \frac{N}{2}$ . If the initial data was assumed to have some symmetric properties,

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Jiang and Zhang [13, 14] obtained the global weak solutions for any  $\gamma > 1$ . For the full Navier-Stokes equations, Feireisl and Petzeltová established the variational solutions, see [10], for example.

Concerning the large time behavior of solutions to the initial-boundary value problem (1.1)-(1.3), by using the weak convergence method, Feireisl and Petzeltová [7] proved the weak solutions to the problem (1.1)-(1.3) with a gradient external force  $\nabla F$  independent of time  $t$  converge to stationary solution  $(\rho_s, 0)$  in the following sense

$$\rho(t) \rightarrow \rho_s \text{ strongly in } L^\gamma(\Omega), \text{ ess sup}_{\tau > t} \int_{\Omega} \rho(\tau) |u(\tau)|^2 dx \rightarrow 0, \text{ as } t \rightarrow \infty,$$

where the domain  $\Omega$  need not to be bounded and the initial data need not to be close to the equilibrium state. If the initial data is close to the equilibrium state, there are many results on the problem of large time behavior of global smooth solutions to the compressible Navier-Stokes equations (of heat-conducting flow). When there is no external or internal force involved, the  $H^s$  global existence and time-decay rate of strong solutions are obtained in whole space  $\mathbb{R}^3$  first by Matsumura and Nishida [21, 22] and the optimal  $L^p$  ( $p \geq 2$ ) decay rate is established by Ponce [25]. The large time decay rate of global solution in multi-dimensional half space or exterior domain is also investigated for the compressible Navier-Stokes equations by Kagei and Kobayashi [15, 16], Kobayashi and Shibata [17], and Kobayashi [18]. Therein, the optimal  $L^2$  time-decay rate in three dimension is established as

$$\|(\rho - \tilde{\rho}, u)(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}},$$

with  $(\tilde{\rho}, 0)$  the constant state, under small initial perturbation in Sobolev space. When additional (exterior or internal) potential force is taken into account, the global existence of a strong solution and convergence to steady state are investigated by Matsumura and Nishida [24] and many other authors [3, 4, 26, 27, 30]. The optimal  $L^p$  convergence rate in  $\mathbb{R}^3$  is established by Duan et al. [6] for the non-isentropic compressible flow as

$$\|(\rho - \tilde{\rho}, u, \theta - \theta_\infty)(t)\|_{L^p(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \quad 2 \leq p \leq 6,$$

where  $(\tilde{\rho}, 0, \theta_\infty)$  is related to the steady-state solution, under the same smallness assumptions on initial perturbation and the external force. If  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^3$ , based on high order energy estimates, Matsumura and Nishida [23] proved that, for large times, the solution decays exponentially to a unique equilibrium state. However, for the one dimensional case, the smallness assumption on the initial data and force can be removed. Indeed, by constructing suitable Lyapunov functionals, the decay rate estimates in  $L^2$  norm and  $H^1$  norm are established by Straškraba and Zlotnik [28], and the decay is exponential if so the decay rate of the nonstationary part of the mass force is.

In this paper, under the hypothesis that  $\rho$  is upper bounded, we construct a Lyapunov functional for the multidimensional isentropic compressible Navier-Stokes equations (1.1) with the aid of the operator  $\mathcal{B}$  introduced by Bogovskii [1] (cf. Lemma 2.5). Based on this, we show that the weak solutions to problem (1.1) decay exponentially to the equilibrium state in  $L^2$  norm. The ideas mainly come from [28], however, unlike [28], we do not divide by  $\rho$  on both side of (1.1)<sub>2</sub>. As a result, the uniformly positive lower bound of  $\rho$  is not necessary in our analysis and vacuum may be admitted. Compared with [23], our analysis follows the framework of Lions [20] and Feireisl et al. [9], and thus the higher regularity of  $(\rho, u)$  is not necessary here. Actually, the upper bound of the density  $\rho$  plays the essential role in our proof. Coincidentally, a blow-up criterion for the 3D compressible Navier-Stokes equations was given in terms of the upper bound of the density  $\rho$  by Sun et al. [29], however, their result does not contain the case for spatial dimension  $N > 3$ .

Now we give a precise formulation of our result. Let  $\rho_s$  be the solution of the following stationary problem:

$$\begin{cases} \nabla P(\rho_s) = 0, \\ \int_{\Omega} \rho_s dx = \int_{\Omega} \rho_0 dx. \end{cases} \quad (1.4)$$

Then  $\rho_s = \frac{1}{|\Omega|} \int_{\Omega} \rho_0 dx$  be a positive constant.

Formally, the total energy of problem (1.1) can be written as

$$E(t) = \int_{\Omega} \frac{1}{2} \rho(t) |u(t)|^2 + \frac{1}{\gamma - 1} \rho^{\gamma}(t) dx,$$

satisfying the energy inequality

$$\frac{dE}{dt} + \int_{\Omega} \mu |\nabla u|^2 + (\lambda + \mu) (\operatorname{div} u)^2 dx \leq 0. \quad (1.5)$$

The definition of weak solutions to the problem (1.1)-(1.3) is given as follows:

**Definition 1.1 (finite energy weak solutions, [20, 9]).** A pair of function  $(\rho, u)$  will be termed a finite energy weak solution of the problem (1.1), (1.3) on  $(0, \infty) \times \Omega$ , if

- $\rho \geq 0, \rho \in L_{loc}^{\infty}(0, \infty; L^{\gamma}(\Omega)), u \in L_{loc}^2(0, \infty; W_0^{1,2}(\Omega))$ .
- The equations (1.1) are satisfied in  $\mathcal{D}'((0, \infty) \times \Omega)$ ; moreover,  $(1.1)_1$  holds in  $\mathcal{D}'((0, \infty) \times \mathbb{R}^N)$  provided  $\rho, u$  were prolonged to be zero on  $\mathbb{R}^N \setminus \Omega$ .
- The energy inequality (1.4) holds in  $\mathcal{D}'(0, \infty)$ .
- The equality  $(1.1)_1$  holds in the sense of renormalized solutions, more precisely, the following equation

$$b(\rho)_t + \operatorname{div}(b(\rho)u) + (b'(\rho)\rho - b(\rho))\operatorname{div} u = 0$$

holds in  $\mathcal{D}'((0, \infty) \times \Omega)$  for any  $b \in C^1(\mathbb{R})$  such that

$$b'(z) = 0 \text{ for all } z \in \mathbb{R} \text{ large enough, say, } |z| \geq M,$$

where the constant  $M$  may vary for different functions  $b$ .

The paper is mainly concerned with the proof of the following theorem.

**Theorem 1.1.** *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^N$ ,  $N \geq 2$ ,  $(\rho, u)$  be a finite energy weak solution to the problem (1.1)-(1.3) and  $\rho_s$  be the solution to the stationary problem (1.4). In addition, if we assume  $\rho$  is upper bounded, i.e., there exists constant  $\bar{\rho} > 0$ , such that*

$$\rho \leq \bar{\rho}, \text{ a.e. } (t, x) \in (0, \infty) \times \Omega$$

Then

$$\int_{\Omega} \rho |u|^2 + (\rho - \rho_s)^2 dx \leq C(E_0, \bar{\rho}) \exp\{-C(\bar{\rho}, \Omega)t\} \text{ for a.e. } t \in (0, \infty), \quad (1.6)$$

where

$$E_0 = \int_{\Omega} \frac{|m_0|^2}{2\rho_0} + \frac{1}{\gamma - 1} \rho_0^{\gamma} dx.$$

*Remark 1.1.* A natural question is that whether the solution  $(\rho, u)$  stated in Theorem 1.1 exists. Indeed, Matsumura and Nishida [23] obtained the existence of global solution to the compressible heat-conductive fluid in bounded domain in  $\mathbb{R}^3$  provided the initial data is close to the equilibrium state. If vacuum is taken into account, Huang, Li and Xin [12] established the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible Navier-Stokes equations in three spatial dimensions with smooth initial data which are of small energy but possibly large oscillations with constant state as far field which could be either vacuum or non-vacuum.

*Remark 1.2.* We believe that our method can be adapted to the other related models. This is the object of our future work.

**Notations:**

1.  $\eta_{\epsilon}(\cdot) = \frac{1}{\epsilon^N} \eta(\frac{\cdot}{\epsilon})$ , where  $\eta$  is the standard mollifier in  $\mathbb{R}^N$ .
2.  $[f]_{\epsilon} = \eta_{\epsilon} * f$ , for any  $f \in L_{loc}^1(\mathbb{R}^N)$ .

The rest of the paper is organized as follows: In section 2, we present some preliminary results which will be used later. In section 3, we give the proof of Theorem 1.1.

## 2 Preliminaries

**Lemma 2.1** ([9]). *Let  $\rho, u$  be a solution of (1.1)<sub>1</sub> in  $\mathcal{D}'((0, \infty) \times \Omega)$  and such that  $\rho \in L^2((0, \infty) \times \Omega)$  and  $u \in L^2(0, \infty; [W_0^{1,2}(\Omega)]^N)$ .*

*Then, prolonging  $\rho, u$  to be zero on  $\mathbb{R}^N \setminus \Omega$ , the equation (1.1)<sub>1</sub> holds in  $\mathcal{D}'((0, \infty) \times \mathbb{R}^N)$ .*

**Lemma 2.2** ([9]). *Let  $(\rho, u)$  be a finite energy weak solution of problem (1.1)-(1.3) on the time interval  $(0, \infty)$ .*

*Then the total mass  $m[\rho(t)] \doteq \int_{\Omega} \rho(t) dx$  is conserved, i.e.,*

$$\int_{\Omega} \rho(t) dx = \int_{\Omega} \rho_0 dx, \quad (2.1)$$

*for all  $t \in (0, \infty)$ .*

**Lemma 2.3** ([19, 10]). *Let  $\Omega \subset \mathbb{R}^N$  be a domain and  $\rho \in L^p(\Omega), u \in [W^{1,q}(\Omega)]^N$  be given functions with  $1 \leq p, q \leq \infty$  and  $\frac{1}{p} + \frac{1}{q} \leq 1$ .*

*Then for any compact  $K \subset \Omega$ ,*

(i)

$$\|[\operatorname{div}(\rho u)]_{\epsilon} - \operatorname{div}([\rho]_{\epsilon} u)\|_{L^r(K)} \leq c(K) \|\rho\|_{L^p(\Omega)} \|u\|_{W^{1,q}(\Omega)} \quad (2.2)$$

*provided  $\epsilon$  is small enough, where  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ . In addition, if  $\Omega = \mathbb{R}^N$ ,  $K$  can be replaced by  $\mathbb{R}^N$ .*

(ii)

$$[\operatorname{div}(\rho u)]_{\epsilon} - \operatorname{div}([\rho]_{\epsilon} u) \rightarrow 0 \text{ in } L^{\theta}(K) \text{ as } \epsilon \rightarrow 0, \quad (2.3)$$

*where  $\frac{1}{\theta} = \frac{1}{p} + \frac{1}{q}$ , if  $p < \infty$  and  $1 \leq \theta < q$  if  $p = \infty$ . In addition, if  $\Omega = \mathbb{R}^N$  and  $p < \infty$ ,  $K$  can be replaced by  $\mathbb{R}^N$ .*

*Proof.* Since the proof of most of the results in this lemma can be found in [19, 10], here we only prove (ii) for the case  $p = \infty$ . To this end, we define  $G_{\epsilon}(\rho) = [\operatorname{div}(\rho u)]_{\epsilon} - \operatorname{div}([\rho]_{\epsilon} u)$ . Choosing any open subset  $U$  such that  $K \subset U \subset \subset \Omega$ , then  $\rho \in L^{\infty}(\Omega)$  implies  $\rho \in L^{\tilde{p}}(U)$ , where  $\tilde{p} = \frac{q\theta}{q-\theta}$  satisfying  $\frac{1}{\tilde{p}} + \frac{1}{q} = \frac{1}{\theta}$ . It is easy to see that  $G_{\epsilon}(\rho) \rightarrow 0$  in  $L^{\theta}(\Omega)$  as  $\epsilon \rightarrow 0$  for any  $\rho \in C_0^{\infty}(\Omega)$ . Now choosing a sequence  $\rho_n \in C_0^{\infty}(U)$  such that  $\rho_n \rightarrow \rho$  in  $L^{\tilde{p}}(U)$  as  $n \rightarrow \infty$ , using the result in (i) with  $\rho, p, r$  and  $\Omega$  replaced by  $\rho - \rho_n, \tilde{p}, \theta$  and  $U$ , respectively, we have

$$\begin{aligned} \|G_{\epsilon}(\rho)\|_{L^{\theta}(K)} &\leq \|G_{\epsilon}(\rho - \rho_n)\|_{L^{\theta}(K)} + \|G_{\epsilon}(\rho_n)\|_{L^{\theta}(K)} \\ &\leq c(K) \|\rho - \rho_n\|_{L^{\tilde{p}}(U)} \|u\|_{W^{1,q}(U)} + \|G_{\epsilon}(\rho_n)\|_{L^{\theta}(K)} \\ &\leq c(K) \|\rho - \rho_n\|_{L^{\tilde{p}}(U)} \|u\|_{W^{1,q}(\Omega)} + \|G_{\epsilon}(\rho_n)\|_{L^{\theta}(\Omega)} \\ &\rightarrow 0, \text{ as } \epsilon \rightarrow 0, n \rightarrow \infty. \end{aligned}$$

This completes the proof of Lemma 2.3. □

**Corollary 2.1.** *If  $\rho \in L^{\infty}((0, \infty) \times \Omega)$  and  $u \in L^2(0, \infty; [W_0^{1,2}(\Omega)]^N)$  solves (1.1)<sub>1</sub> in  $\mathcal{D}'((0, \infty) \times \Omega)$ .*

*Then for any subset  $[\alpha, \beta] \subset (0, \infty)$ , we have*

$$\partial_t [\rho]_{\epsilon} + \operatorname{div}([\rho]_{\epsilon} u) = r_{\epsilon} \text{ a.e. on } [\alpha, \beta] \times \Omega, \quad (2.4)$$

*where  $r_{\epsilon} = \operatorname{div}([\rho]_{\epsilon} u) - [\operatorname{div}(\rho u)]_{\epsilon}$ . Moreover,  $r_{\epsilon}$  is bounded in  $L^2([\alpha, \beta] \times \Omega)$  uniformly in  $\epsilon$  and converges to 0 strongly in  $L^2(\alpha, \beta; L^{\theta}(\Omega))$  for all  $\theta \in [1, 2)$ .*

*Proof.* Firstly, by virtue of Lemma 2.1, the equation (1.1)<sub>1</sub> holds in  $\mathcal{D}'((0, \infty) \times \mathbb{R}^N)$  provided  $\rho, u$  were extended to be zero on  $\mathbb{R}^N \setminus \Omega$ . Then we use the mollifier  $\eta_{\epsilon}$  as test functions to deduce (2.4) provided  $\epsilon > 0$  is small enough. It follows from Lemma 2.3 immediately that  $r_{\epsilon}$  is bounded in  $L^2([\alpha, \beta] \times \Omega)$  uniformly in  $\epsilon$ , together with Lebesgue's dominated convergence theorem, we have  $r_{\epsilon} \rightarrow 0$  in  $L^2(\alpha, \beta; L^{\theta}(\Omega))$  for  $\theta \in [1, 2)$ , where we have used the fact that  $\Omega$  is bounded. □

**Lemma 2.4** ([8, 9]). *Let  $\Omega$  be a bounded Lipschitz domain in  $\mathbb{R}^N$ ,  $N \geq 2$ , and  $p, r \in (1, \infty)$  given numbers,  $f \in \{L^p(\Omega) \mid \int_{\Omega} f dx = 0\}$ .*

*Then the problem*

$$\operatorname{div} v = f, v|_{\partial\Omega} = 0, \quad (2.5)$$

*admits a solution operator  $\mathcal{B} : f \mapsto v$  enjoying the following properties:*

- *$\mathcal{B}$  is a linear operator from  $L^p(\Omega)$  into  $[W_0^{1,p}(\Omega)]^N$ , i.e.,*

$$\|\mathcal{B}[f]\|_{W^{1,p}(\Omega)} \leq c(p, \Omega) \|f\|_{L^p(\Omega)};$$

- *The function  $v = \mathcal{B}[f]$  solves the problem (2.5);*
- *If a function  $f \in L^p(\Omega)$  can be written in the form  $f = \operatorname{div} g$  with  $g \in [L^r(\Omega)]^N$  and  $g \cdot n = 0$  on  $\partial\Omega$ , where  $n$  is the outward pointing unit normal vector field along  $\partial\Omega$ , then*

$$\|\mathcal{B}[f]\|_{L^r(\Omega)} \leq c(p, r, \Omega) \|g\|_{L^r(\Omega)}.$$

*Remark 2.1.* To our best knowledge, the operator  $\mathcal{B}[\cdot]$  was first constructed by Bogovskii[1]. A complete proof of the above mentioned properties may be found in Galdi [11] or Borchers and Sohr[2]. Moreover, the operator  $\mathcal{B}[\cdot]$  was first used by Feireisl and Petzeltová [8] to show the existence of weak solutions  $(\rho, u)$  to the problem (1.1)-(1.3) with the density  $\rho$  square integrable up to the boundary  $\partial\Omega$ .

### 3 proof of theorem 1.1

**Lemma 3.1.** *Let  $r_0 > 0, \bar{r} > 0$  and  $\gamma > 1$  be arbitrary fixed constants,  $f(r) = r \int_{r_0}^r \frac{h^\gamma - r_0^\gamma}{h^2} dh$  for  $r \in [0, \bar{r}]$ . Then there exists positive constants  $K_1$  and  $K_2$  depending on  $r_0$  and  $\bar{r}$ , such that*

$$K_1(r - r_0)^2 \leq f(r) \leq K_2(r - r_0)^2 \quad \text{for all } r \in [0, \bar{r}]. \quad (3.1)$$

*Proof.* Let

$$g(r) = \frac{r \int_{r_0}^r \frac{h^\gamma - r_0^\gamma}{h^2} dh}{(r - r_0)^2}.$$

It is easy to see that

$$\lim_{r \rightarrow 0} g(r) = \frac{\lim_{r \rightarrow 0} r \int_{r_0}^r \frac{h^\gamma - r_0^\gamma}{h^2} dh}{r_0^2} = r_0^{\gamma-2} > 0,$$

Using the l'Hospital rule, we obtain

$$\lim_{r \rightarrow r_0} g(r) = \lim_{r \rightarrow r_0} \frac{\int_{r_0}^r \frac{h^\gamma - r_0^\gamma}{h^2} dh + \frac{r^\gamma - r_0^\gamma}{r}}{2(r - r_0)} = \frac{\gamma}{2} r_0^{\gamma-2} > 0.$$

Consequently,  $g(r)$  is a continuous function on  $[0, \bar{r}]$  with  $g(r) > 0$ , and (3.1) follows immediately.  $\square$

Now we are going to give the proof of Theorem 1.1. First of all, we need to rewrite the energy inequality (1.5) as a new form. To this end, we choose an arbitrary  $\psi(t) \in \mathcal{D}(0, \infty)$  with  $\psi(t) \geq 0$ , then the energy inequality (1.5) is equivalent to

$$-\int_0^\infty \psi_t \int_{\Omega} \frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma-1} dx dt + \int_0^\infty \psi \int_{\Omega} \mu |\nabla u|^2 + (\lambda + \mu) (\operatorname{div} u)^2 dx dt \leq 0.$$

Recalling that  $\rho_s = \frac{1}{|\Omega|} \int_{\Omega} \rho_0 dx$  is a positive constant and  $\int_{\Omega} \rho(t) dx$  is independent of  $t$  due to Lemma 2.2, we thus have

$$\int_0^{\infty} \psi_t \int_{\Omega} \left( -\frac{\gamma}{\gamma-1} \rho \rho_s^{\gamma} + \rho_s^{\gamma} \right) dx dt = 0.$$

Adding the above two equations, we have

$$-\int_0^{\infty} \psi_t \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \rho \int_{\rho_s}^{\rho} \frac{h^{\gamma} - \rho_s^{\gamma}}{h^2} dh \right) dx dt + \int_0^{\infty} \psi \int_{\Omega} (\mu |\nabla u|^2 + (\lambda + \mu) (\operatorname{div} u)^2) dx dt \leq 0. \quad (3.2)$$

Next, we use the operator  $\mathcal{B}$  introduced in Lemma 2.4 to construct test function of the form

$$\Phi(t, x) = \psi(t) \mathcal{B}[[\rho]_{\epsilon} - \rho_s^{\epsilon}],$$

where  $\psi$  is the same as in (3.2) and  $\rho_s^{\epsilon} = \frac{1}{|\Omega|} \int_{\Omega} [\rho]_{\epsilon} dx$ . Obviously,  $\Phi(t, x)$  is smooth in  $x$  and vanishes near  $\partial\Omega$  due to the properties of operator  $\mathcal{B}$ . Moreover, since  $\rho \in L^{\infty}(0, \infty; \Omega)$ ,  $\Phi_t$  is in  $L^2(0, \infty; [W_0^{1,2}(\Omega)]^N)$  in view of Corollary 2.1. Consequently,  $\Phi$  could be used as a test function for the equation (1.1)<sub>2</sub>. Thus, we have

$$\begin{aligned} & -\int_0^{\infty} \psi_t \int_{\Omega} \rho u \mathcal{B}[[\rho]_{\epsilon} - \rho_s^{\epsilon}] dx dt + \int_0^{\infty} \psi \int_{\Omega} \rho u \mathcal{B}[\operatorname{div}([\rho]_{\epsilon} u)] dx dt \\ & -\int_0^{\infty} \psi \int_{\Omega} \rho u \mathcal{B}[r_{\epsilon} - \frac{1}{|\Omega|} \int_{\Omega} r_{\epsilon} dx] dx dt - \int_0^{\infty} \psi \int_{\Omega} \rho u \otimes u : \nabla \mathcal{B}[[\rho]_{\epsilon} - \rho_s^{\epsilon}] dx dt \\ & -\int_0^{\infty} \psi \int_{\Omega} (P(\rho) - P(\rho_s)) ([\rho]_{\epsilon} - \rho_s^{\epsilon}) dx dt + \mu \int_0^{\infty} \psi \int_{\Omega} \nabla u : \nabla \mathcal{B}[[\rho]_{\epsilon} - \rho_s^{\epsilon}] dx dt \\ & + (\lambda + \mu) \int_0^{\infty} \psi \int_{\Omega} \operatorname{div} u ([\rho]_{\epsilon} - \rho_s^{\epsilon}) dx dt = 0, \end{aligned} \quad (3.3)$$

where we have used Corollary 2.1. Letting  $\epsilon \rightarrow 0$ , we obtain by virtue of Corollary 2.1 and the properties of operator  $\mathcal{B}$ ,

$$\begin{aligned} & -\int_0^{\infty} \psi_t \int_{\Omega} \rho u \mathcal{B}[\rho - \rho_s] dx dt + \int_0^{\infty} \psi \int_{\Omega} \rho u \mathcal{B}[\operatorname{div}(\rho u)] dx dt \\ & -\int_0^{\infty} \psi \int_{\Omega} \rho u \otimes u : \nabla \mathcal{B}[\rho - \rho_s] dx dt \\ & -\int_0^{\infty} \psi \int_{\Omega} (P(\rho) - P(\rho_s)) (\rho - \rho_s) dx dt + \mu \int_0^{\infty} \psi \int_{\Omega} \nabla u : \nabla \mathcal{B}[\rho - \rho_s] dx dt \\ & + (\lambda + \mu) \int_0^{\infty} \psi \int_{\Omega} \operatorname{div} u (\rho - \rho_s) dx dt = 0, \end{aligned} \quad (3.4)$$

Multiplying (3.4) by a negative constant  $-\sigma$  with  $0 < \sigma \ll 1$  and summing up the resulting equation with the energy inequality (3.2), we get

$$\begin{aligned} & -\int_0^{\infty} \psi_t \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \rho \int_{\rho_s}^{\rho} \frac{h^{\gamma} - \rho_s^{\gamma}}{h^2} dh - \sigma \rho u \mathcal{B}[\rho - \rho_s] \right) dx dt \\ & + \int_0^{\infty} \psi \int_{\Omega} (\mu |\nabla u|^2 + (\lambda + \mu) (\operatorname{div} u)^2) dx dt - \sigma \int_0^{\infty} \psi \int_{\Omega} \rho u \mathcal{B}[\operatorname{div}(\rho u)] dx dt \\ & + \sigma \int_0^{\infty} \psi \int_{\Omega} \rho u \otimes u : \nabla \mathcal{B}[\rho - \rho_s] dx dt + \sigma \int_0^{\infty} \psi \int_{\Omega} (\rho^{\gamma} - \rho_s^{\gamma}) (\rho - \rho_s) dx dt \\ & - \sigma \mu \int_0^{\infty} \psi \int_{\Omega} \nabla u : \nabla \mathcal{B}[\rho - \rho_s] dx dt - \sigma (\lambda + \mu) \int_0^{\infty} \psi \int_{\Omega} \operatorname{div} u (\rho - \rho_s) dx dt \leq 0. \end{aligned} \quad (3.5)$$

Let

$$V_{\sigma} = \int_{\Omega} \left( \frac{1}{2} \rho |u|^2 + \rho \int_{\rho_s}^{\rho} \frac{h^{\gamma} - \rho_s^{\gamma}}{h^2} dh - \sigma \rho u \mathcal{B}[\rho - \rho_s] \right) dx,$$

and

$$\begin{aligned}
W_\sigma = & \int_{\Omega} (\mu |\nabla u|^2 + (\lambda + \mu)(\operatorname{div} u)^2) dx - \sigma \int_{\Omega} \rho u \mathcal{B}[\operatorname{div}(\rho u)] dx \\
& + \sigma \int_{\Omega} \rho u \otimes u : \nabla \mathcal{B}[\rho - \rho_s] dx + \sigma \int_{\Omega} (\rho^\gamma - \rho_s^\gamma)(\rho - \rho_s) dx \\
& - \sigma \mu \int_{\Omega} \nabla u : \nabla \mathcal{B}[\rho - \rho_s] dx - \sigma(\lambda + \mu) \int_{\Omega} \operatorname{div} u (\rho - \rho_s) dx.
\end{aligned}$$

Using the fact  $\rho \leq \bar{\rho}$  and the properties of operator  $\mathcal{B}$ , we have

$$\left| \int_{\Omega} -\sigma \rho u \mathcal{B}[\rho - \rho_s] dx \right| \leq \frac{\sigma}{2} \int_{\Omega} \rho |u|^2 dx + \frac{\sigma \bar{\rho} c(\Omega)}{2} \int_{\Omega} (\rho - \rho_s)^2 dx. \quad (3.6)$$

Then it follows from (3.6) and Lemma 3.1 that

$$c_0(\sigma, \bar{\rho}, \Omega) \int_{\Omega} \rho |u|^2 + (\rho - \rho_s)^2 dx \leq V_\sigma \leq c_1(\sigma, \bar{\rho}, \Omega) \int_{\Omega} |u|^2 + (\rho - \rho_s)^2 dx, \quad (3.7)$$

provided  $\sigma$  is small enough.

On the other hand, by Hölder's inequality, we have

$$\begin{aligned}
\left| -\sigma \int_{\Omega} \rho u \mathcal{B}[\operatorname{div}(\rho u)] dx \right| & \leq \sigma \|\rho u\|_{L^2(\Omega)} \|\mathcal{B}[\operatorname{div}(\rho u)]\|_{L^2(\Omega)} \\
& \leq \sigma c(\Omega) \|\rho u\|_{L^2(\Omega)}^2 \leq \sigma c(\Omega) \bar{\rho}^2 \|u\|_{L^2(\Omega)}^2, \\
\left| \sigma \int_{\Omega} \rho u \otimes u : \nabla \mathcal{B}[\rho - \rho_s] dx \right| & \leq \sigma \bar{\rho} \left( \int_{\Omega} |u|^{2p} dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |\nabla \mathcal{B}[\rho - \rho_s]|^q dx \right)^{\frac{1}{q}} \\
& \leq \sigma \bar{\rho} c(\Omega) \int_{\Omega} |\nabla u|^2 dx \left( \int_{\Omega} |\rho - \rho_s|^q dx \right)^{\frac{1}{q}} \\
& \leq \sigma \bar{\rho}^2 c(\Omega) \int_{\Omega} |\nabla u|^2 dx,
\end{aligned}$$

where we take  $p = \begin{cases} \frac{N}{N-2} & \text{if } N \geq 3, \\ 2 & \text{if } N = 2 \end{cases}$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

$$\begin{aligned}
\left| -\sigma \mu \int_{\Omega} \nabla u : \nabla \mathcal{B}[\rho - \rho_s] dx \right| & \leq \frac{\mu}{4} \int_{\Omega} |\nabla u|^2 dx + \sigma^2 \mu \int_{\Omega} |\nabla \mathcal{B}[\rho - \rho_s]|^2 dx \\
& \leq \frac{\mu}{4} \int_{\Omega} |\nabla u|^2 dx + \sigma^2 \mu c(\Omega) \int_{\Omega} |\rho - \rho_s|^2 dx, \\
\left| -\sigma(\lambda + \mu) \int_{\Omega} \operatorname{div} u (\rho - \rho_s) dx \right| & \leq \frac{\lambda + \mu}{4} \int_{\Omega} |\operatorname{div} u|^2 dx + \sigma^2(\lambda + \mu) \int_{\Omega} |\rho - \rho_s|^2 dx,
\end{aligned}$$

and it is easy to see that,

$$\sigma \int_{\Omega} (\rho^\gamma - \rho_s^\gamma)(\rho - \rho_s) dx \geq \sigma c \int_{\Omega} |\rho - \rho_s|^2 dx.$$

In view of the above five estimates, we have

$$W_\sigma \geq c_2(\sigma, \bar{\rho}, \Omega) \int_{\Omega} |u|^2 + (\rho - \rho_s)^2 dx, \quad (3.8)$$

provided  $\sigma$  is small enough.

Therefore, from (3.7) and (3.8), we deduce that for appropriate selected  $\sigma \ll 1$ , there exists positive constant  $C(\bar{\rho}, \Omega)$ , such that

$$C(\bar{\rho}, \Omega) V_\sigma \leq W_\sigma. \quad (3.9)$$

Combining (3.9) with (3.5), we obtain

$$-\int_0^\infty \psi_t(t) V_\sigma(t) dt + C(\bar{\rho}, \Omega) \int_0^\infty \psi(t) V_\sigma(t) dt \leq 0, \quad (3.10)$$

for any  $\psi \in \mathcal{D}(0, \infty)$  with  $\psi \geq 0$ .

Let  $[\alpha, \beta]$  be any compact subset of  $(0, \infty)$ , taking  $\psi(t) = \eta_\epsilon(t - \cdot)$  in (3.10), we have

$$\partial_t [V_\sigma]_\epsilon + C(\bar{\rho}, \Omega) [V_\sigma]_\epsilon \leq 0, \text{ a.e. } t \in [\alpha, \beta], \quad (3.11)$$

provided  $\epsilon$  is small enough.

Thus

$$[V_\sigma]_\epsilon(t) \leq [V_\sigma]_\epsilon(s) \exp\{-C(\bar{\rho}, \Omega)(t - s)\}$$

for a.e.  $0 < s < t < \infty$ , according to (3.10). Recalling that  $V_\sigma(t) \in L_{loc}^\infty(0, \infty)$ , letting  $\epsilon \rightarrow 0$ , we have

$$V_\sigma(t) \leq V_\sigma(s) \exp\{-C(\bar{\rho}, \Omega)(t - s)\} \leq C(E_0, \bar{\rho}) \exp\{-C(\bar{\rho}, \Omega)t\}, \quad (3.12)$$

where  $E_0$  denotes the initial energy.

Consequently, (1.6) follows from (3.7) and (3.12) immediately. This completes the proof of Theorem 1.1. □

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